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1974 J. Phys. A: Math. Nucl. Gen. 7 1256

(http://iopscience.iop.org/0301-0015/7/11/005)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 11, 1974. Printed in Great Britain.

Covariant Feynman derivation of Schrödinger's equation in a riemannian space

J S Dowker

Department of Theoretical Physics, The University of Manchester, Manchester M13 9PL, UK

Received 23 January 1974

Abstract. Ruse's covariant Taylor expansion is used to give a neat and rapid derivation of the covariant Schrödinger equation for a particle moving in a riemannian space. It is pointed out that the calculation is basically the same as Kolmogorov's derivation of the diffusion equation, and a covariant form of the stochastic differential equation is given.

1. Introduction

The quantization of a system described by the classical lagrangian

$$L = \tfrac{1}{2} g_{\alpha\beta}(q)\dot{q}^{\alpha}\dot{q}^{\beta}, \qquad \dot{q}^{\alpha} = \frac{\mathrm{d}q^{\alpha}}{\mathrm{d}t}, \qquad \alpha = 1, 2, \ldots, r, \tag{1}$$

ie a free particle on a riemannian space, V_r, of metric $g_{\alpha\beta}$, has been described by De Witt (1957) and lately by others. In particular the Feynman quantization (Feynman 1948) has been considered and we would like to make some further methodological comments on this interesting question.

It seems reasonable that Schrödinger's equation should take the covariant form

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\tfrac{1}{2}\Delta_2\psi \tag{2}$$

where Δ_2 is the (covariant) Laplace–Beltrami (Lamé) operator,

$$\Delta_2 = g^{-1/2}\,\partial_{\alpha}(g^{\alpha\beta}g^{1/2}\,\partial_{\beta})$$

(see eg Podolsky 1928, Kramers 1957, De Witt 1952). Logically there is no reason why the Schrödinger equation should take the particular covariant form (2), nor indeed why it should be covariant at all. The quantization of a classical system is not a logical procedure. Rather, since the world is (we think) a quantum-mechanical one, the logical process runs from the quantum to the (approximate) classical domain. Nevertheless, whilst this attitude is epistemologically correct it is not very helpful. The practical approach seems to be to start from a classical system, say (1), then to guess a Schrödinger equation, say (2), and finally to compare with experiment. Such comparison is really the only justification for the hamiltonian $-\frac{1}{2}\nabla^2 + V$ of elementary quantum mechanics corresponding to the classical lagrangian $\frac{1}{2}\dot{q}^2 - V$. A more pertinent example is that of the spherical top, the lagrangian for which is given by (1), V_r being a three-dimensional sphere, S^3, with antipodal points identified, S^3/Z_2. The quantum mechanics of this

system was first given by Reiche and Rademacher (Reiche and Rademacher 1926, Rademacher and Reiche 1927) and the Schrödinger equation was just that of (2). Experiment justifies this choice. More interestingly (2) is also the form of the Schrödinger equation used for the asymmetrical top, the configuration space of which is a deformed three-sphere. In this case the scalar concomitants of the curvature tensor, eg R, $R_{\alpha\beta}R^{\alpha\beta}$ etc, which can always be added into (2) as potentials, are still constant, as they are for S^3. Therefore no such terms have ever been considered by the professional molecular spectroscopists (eg King 1947).

The Feynman quantization of the system can now be deduced by folding together the short-time (quasi-classical) propagator of (2):

$$\langle q'', t''|q', t'\rangle_c = g''^{-1/4}D^{1/2}(q'', t''|q', t')g'^{-1/4}\exp[iS(q'', t''|q', t')] \tag{3}$$

where

$$S(q'', t''|q', t') = \int_{\substack{t' \\ \text{classical} \\ \text{path}}}^{t''} (L+\tfrac{1}{12}R)\,dt$$

and where D is the van Vleck determinant (see De Witt 1957 for details).

In this manner De Witt (1957) arrived at the formal functional integral representation of the propagator $\langle q'', t''|q', t'\rangle$ of equation (2):

$$\langle q'', t''|q', t'\rangle = \mathcal{N}\int \exp\left(i\int_{t'}^{t''}(L+\tfrac{1}{6}R)\,dt\right)\mathcal{D}[q] \tag{4}$$

where

$$\mathcal{N} = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} (2\pi i\epsilon)^{-\frac{1}{2}r(n+1)}$$

and the invariant measure $\mathcal{D}[q]$ is given by

$$\mathcal{D}[q] = \lim_{\substack{n\to\infty \\ \epsilon\to 0}} \prod_{i=1}^{n} dq(t_i)g_i^{1/2}, \qquad t_{i+1}-t_i = \epsilon. \tag{5}$$

This is a constructional approach to Feynman quantization and it is of interest to reverse the procedure and derive Schrödinger's equation from (4) in the manner of Feynman (1948). Such a calculation is mentioned by De Witt (1957, footnote on p 395) and has been explicitly detailed by Cheng (1972). It is also contained in our earlier work (Mayes and Dowker 1973).

If one starts from the form (4) then there is no *a priori* reason to include the $\frac{1}{6}R$ term, which will then appear in the Schrödinger equation as a potential. One might take this as indicating that there *should* be a $\frac{1}{6}R$ term in (2) and indeed there is some strength in this argument if one believes that the Feynman postulate is more basic than simply writing down a Schrödinger equation and, further, if one believes that this postulate is to be written down in as 'simple' and as 'natural' a way as possible. Thus there is no *a priori* reason why the measure in (4) should be given by (5), but this is the most natural choice.

Bearing these remarks in mind we shall take Feynman's principle in the form of (4) where $\mathcal{N}$ is to be determined and $\mathcal{D}[q]$ is given by (5). Our aim in this paper is to present a somewhat neater and more attractive derivation of (2) from (4) than the ones available (eg Cheng 1972).

2. Derivation of Schrödinger's equation

Basically the calculation is the same as that in Kolmogorov's (1928, 1931) derivation of the backwards and forwards diffusion equations (see eg Lévy 1948, p 66). In this derivation, and in the theory of diffusion in general, important roles are played by the averages†,

$$a_{\mathrm{F}}^{\alpha}(t', q') = \lim_{\epsilon \to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q''(q''^{\alpha} - q'^{\alpha}) \langle q'', t' + \epsilon | q', t' \rangle$$

$$b_{\mathrm{F}}^{\alpha\beta}(t', q') = \lim_{\epsilon \to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q''(q''^{\alpha} - q'^{\alpha})(q''^{\beta} - q'^{\beta}) \langle q'', t' + \epsilon | q', t' \rangle, \tag{6}$$

the drift 'vector' and local variance matrix respectively. We note that in order to avoid duplication we are using the quantum-mechanical notation. In diffusion theory $\langle q'', t'' | q', t' \rangle$ would be a probability density function, and there would be no 'i'.

Conventionally, the averages in (6) are taken over the 'final' variables q'' and we obtain Schrödinger's equation in the form

$$\mathrm{i}\frac{\partial \psi''}{\partial t''} = g''^{-1/2}\, \partial''_{\alpha}(g''^{1/2} a_{\mathrm{F}}''^{\alpha} \psi'') - \tfrac{1}{2} g''^{-1/2}\, \partial''_{\alpha}\, \partial''_{\beta}(g''^{1/2} b_{\mathrm{F}}''^{\alpha\beta} \psi'') \tag{7}$$

(cf Yosida 1948, for forward diffusion equation). For compactness we have put $\psi'' = \psi(q'', t'')$, $a_{\mathrm{F}}''^{\alpha} = a_{\mathrm{F}}^{\alpha}(t'', q'')$ etc.

Alternatively averages over the initial variables q' can be used,

$$a_{\mathrm{I}}^{\alpha}(t'', q'') = \lim_{\epsilon \to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q'(q''^{\alpha} - q'^{\alpha}) \langle q'', t'' | q', t'' - \epsilon \rangle$$

$$b_{\mathrm{I}}^{\alpha\beta}(t'', q'') = \lim_{\epsilon \to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q'(q''^{\alpha} - q'^{\alpha})(q''^{\beta} - q'^{\beta}) \langle q'', t'' | q', t'' - \epsilon \rangle, \tag{8}$$

and then Schrödinger's equation is found to be

$$\mathrm{i}\frac{\partial \psi''}{\partial t''} = a_{\mathrm{I}}''^{\alpha}\, \partial''_{\alpha} \psi'' - \tfrac{1}{2} b_{\mathrm{I}}''^{\alpha\beta}\, \partial''_{\alpha}\, \partial''_{\beta} \psi''. \tag{9}$$

Time reversal invariance actually implies the relations

$$a_{\mathrm{F}}^{\alpha} = -a_{\mathrm{I}}^{\alpha}$$

$$b_{\mathrm{F}}^{\alpha\beta} = b_{\mathrm{I}}^{\alpha\beta}.$$

The fact that no 'potential' term seems to appear in the classical derivations of the diffusion equation is due to the assumption that the total final probability equals unity, ie that

$$\int \langle q'', t'' | q', t' \rangle \, \mathrm{d}q'' = 1.$$

This condition is not valid for quantum mechanics. Rather we should use

$$\lim_{\epsilon \to 0} (\mathrm{i}\epsilon)^{-1} \int \langle q'', t' + \epsilon | q', t' \rangle \, \mathrm{d}q'' = \lim_{\epsilon \to 0} [(\mathrm{i}\epsilon)^{-1} - V(q', t')]. \tag{10}$$

† In this formula $\mathrm{d}q$ is the invariant volume element.

It is easy to show that for the particular expressions

$$b^{\alpha\beta} = g^{\alpha\beta}$$
$$\alpha_F^{\alpha} = \tfrac{1}{2}g^{-1/2}\,\partial_\beta(g^{1/2}g^{\alpha\beta}) = -g^{\beta\gamma}\Gamma^{\alpha}_{\beta\gamma} \tag{11}$$

equations (7) and (9) are the same as (2). Therefore we have only to show that these values, and condition (10) with $V = 0$, do result from the Feynman postulate which gives the specific short-time form

$$\lim_{\epsilon\to 0}\langle q'', t'+\epsilon|q', t'\rangle \sim \mathcal{N}(\epsilon)\exp[\mathrm{i}\epsilon^{-1}\Omega(q'', q') + \mathrm{i}\tfrac{1}{6}R''\epsilon] \tag{12}$$

where Ω is the Ruse–Synge world function related to the geodesic distance s between q'' and q' by $\Omega = \frac{1}{2}s^2$.

Expression (12) is substituted into, say, (6) and the limits calculated by expanding Ω about q' (or q'' if (8) is used). The first term, $\frac{1}{2}\epsilon^{-1}g'_{\alpha\beta}(q''^{\alpha}-q'^{\alpha})(q''^{\beta}-q'^{\beta})$, is kept in the exponential and the remainder expanded in a power series. This allows one to use the standard expressions for the moments of a gaussian and the result is finally equation (2). The calculation is a little involved algebraically essentially because the expansions used are not covariant, being expansions in $(q''^{\alpha}-q'^{\alpha})$. Our modest aim here is simply to present a neater, covariant calculation which results when one uses the covariant Taylor series of Ruse (1931) described in the next section.

3. Covariant Taylor theorem

Take two non-singular points of V_r, q'' and q', and require the value of the scalar function $\phi(q)$ at q' in terms of quantities at q'' (or vice versa). Ruse (1931) shows that

$$\phi(q') = \phi(q'') - \Omega^{\alpha''}\phi_{,\alpha''} + \tfrac{1}{2}\Omega^{\alpha''}\Omega^{\beta''}\phi_{,\alpha''\beta''} - \ldots \tag{13}$$

where Ω^{α} is just the derivative

$$\Omega^{\alpha''} \equiv g^{\alpha\beta}(q'')\frac{\partial\Omega(q'', q')}{\partial q''^{\beta}} = g''^{\alpha\beta}\,\partial''_{\beta}\Omega$$

and the object $\phi_{,\alpha_1\ldots\alpha_m}$ is the mth (affine) extension of ϕ (Veblen 1927, chap 6, Veblen and Thomas 1923) and equals the mth covariant derivative plus, in general, terms depending on the riemannian curvature. It is a tensor and so the expansion (13) is a covariant one.

For our purposes at the present we need only know that the second affine extension $\phi_{,\alpha\beta}$ coincides with the second covariant derivative $\phi_{\|\alpha\beta} = \nabla_\alpha\nabla_\beta\phi$. Thus we have

$$\phi(q') = \phi(q'') - \Omega^{\alpha''}\phi''_{\|\alpha} + \tfrac{1}{2}\Omega^{\alpha''}\Omega^{\beta''}\phi''_{\|\alpha\beta} + \ldots \tag{14}$$

and, interchanging q' and q'',

$$\phi(q'') = \phi(q') - \Omega^{\alpha'}\phi'_{\|\alpha} + \tfrac{1}{2}\Omega^{\alpha'}\Omega^{\beta'}\phi'_{\|\alpha\beta} - \ldots. \tag{15}$$

Further remarks on the expansion (13) will be found in § 5.

4. Covariant derivation of Schrödinger's equation

The calculation proceeds exactly as in Kolmogorov (1931) or Feynman (1948). We write

$$i\frac{\partial\psi''}{\partial t''} = \lim_{\epsilon\to 0} i\epsilon^{-1}\left(\int \langle q'', t''|q', t''-\epsilon\rangle\psi(q', t'')\,dq' - \psi(q'', t'')\right) \tag{16}$$

and seek to use the expansion (14) applied to ψ. Flrstly, however, we consider the condition (10). If we use the short-time form (12) we find for the left-hand side of (10)

$$\lim_{\epsilon\to 0}(i\epsilon)^{-1}\int \mathcal{N}(\epsilon)\exp(i\epsilon^{-1}\Omega + i\tfrac{1}{6}R''\epsilon)\,dq' \tag{17}$$

where we should now state that for convenience we are now averaging over the initial variables q'.

The advantage of using the Ω^α as expansion variables is that we can use the important formulae for Ω:

$$\Omega = \tfrac{1}{2}g''_{\alpha\beta}\Omega^{\alpha''}\Omega^{\beta''} \tag{18}$$

$$= \tfrac{1}{2}g'_{\alpha\beta}\Omega^{\alpha'}\Omega^{\beta'} \tag{19}$$

which can be thought of as the expansions for Ω. In view of these formulae it is clear that we should transform integration variables in (17) from the q'^α to the $\Omega^{\alpha''}$. If this is done we find

$$\lim_{\epsilon\to 0}(i\epsilon)^{-1}\mathcal{N}(\epsilon)\int \exp\left(\frac{i}{2\epsilon}g''_{\alpha\beta}\Omega^{\alpha''}\Omega^{\beta''} + i\tfrac{1}{6}R''\epsilon\right) g''^{1/2}\Delta^{-1}\,d\Omega'' \tag{20}$$

with $d\Omega'' = \Pi_\alpha\, d\Omega^{\alpha''}$, where the biscalar $\Delta(q'', q')$ equals $g''^{-1/2}Dg'^{-1/2}$ and $D = \det\|-\Omega_{\alpha''\alpha'}\|$. Before the limit can be evaluated Δ must be expanded about q''. This expansion reads

$$\Delta(q'', q') = [\Delta(q'', q') - \Omega^{\alpha''}\Delta_{\|\alpha''} + \tfrac{1}{2}\Omega^{\alpha''}\Omega^{\beta''}\Delta_{\|\alpha''\beta''} - \ldots]_{q'\to q''}.$$

Now the coincidence limits of Δ, $\Delta_{\|\alpha}$ and $\Delta_{\|\alpha\beta}$ have been determined by De Witt (1964) by a purely covariant method which involves only the Ricci identity. He finds

$$\Delta|_{q'\to q''} = 1, \qquad \Delta_{\|\alpha''}|_{q'\to q''} = 0, \qquad \Delta_{\|\alpha''\beta''}|_{q'\to q''} = +\tfrac{1}{3}R_{\alpha\beta}(q'').$$

Thus†

$$\Delta^{-1}(q'', q') = 1 - \tfrac{1}{6}\Omega^{\alpha''}\Omega^{\beta''}R_{\alpha\beta}(q'') + \ldots. \tag{21}$$

This series is substituted into (20) and the integrals performed using the well known expressions

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left(\frac{i}{2\epsilon}g_{\alpha\beta}x^\alpha x^\beta\right)\prod_1^r dx^\alpha = (2\pi i\epsilon)^{r/2}g^{-1/2} \tag{22}$$

$$\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}x^\delta x^\gamma\exp\left(\frac{i}{2\epsilon}g_{\alpha\beta}x^\alpha x^\beta\right)\prod_1^r dx^\alpha = (2\pi i\epsilon)^{r/2}g^{-1/2}g^{\delta\gamma}i\epsilon. \tag{23}$$

Higher moments give higher powers of ϵ on the right-hand side and this is why higher terms in the above expansions do not contribute in the limit of vanishing ϵ.

† Alternatively we could have used the expansion of Ruse's invariant ρ, $= \Delta^{-1}$, developed by Walker (1942).

It is easy to see that the $\frac{1}{6}R$ term coming from the expansion of the exponential will cancel the $\frac{1}{6}R$ term arising from (21) and (23). Also the first term in (21), with (22), yields the standard normalization $\mathcal{N}(\epsilon) = (2\pi i\epsilon)^{-r/2}$ and so we have

$$\lim_{\epsilon\to 0}(i\epsilon)^{-1}\int \langle q'', t''+\epsilon|q', t''\rangle \, dq' = \lim_{\epsilon\to 0}(i\epsilon)^{-1}$$

which now allows us to rewrite equation (16) as

$$i\frac{\partial\psi''}{\partial t''} = \lim_{\epsilon\to 0} i\mathcal{N}(\epsilon)\epsilon^{-1}\int \exp\left(\frac{i}{2\epsilon}g''_{\alpha\beta}\Omega^{\alpha''}\Omega^{\beta''}+i\frac{1}{6}R''\epsilon\right)\left(\frac{1}{2}\Omega^{\alpha''}\Omega^{\beta''}\psi_{\|\alpha''\beta''}-\Omega^{\alpha''}\psi_{\|\alpha''}\right)$$
$$\times g''^{-1/2}\Delta^{-1}\,d\Omega''$$

where we have used the expansion (14) for ψ and have transformed variables as before. Application of (21) and (23) quickly yields the Schrödinger equation

$$i\frac{\partial\psi''}{\partial t''} = -\tfrac{1}{2}g''^{\alpha\beta}\psi''_{\|\alpha\beta} = -\tfrac{1}{2}\Delta''_2\psi''$$

as expected. Note that the Laplace–Beltrami operator appears directly and not piecemeal, as in the calculations referred to earlier (cf Cheng 1972). With this result we have achieved our object of presenting a covariant calculation. The chief advantage is that there is no need to expand the action to fourth order in $(q''-q')$ and, therefore, no need for the quartic moments of a gaussian. This leads to a considerable saving in labour.

We should now like to make a few comments on the preceding calculations the first being technical in character and concerns the expansion given in § 3.

5. Comments

The covariant expansion (13) is derived by Ruse (1931) by first working in normal coordinates and then transforming to an arbitrary system (cf also Ruse 1932). Clearly in our calculation we could have left this last step until the very end. This is probably a more efficient method in that one doesn't have to develop more than one needs. In fact the only expansion required is that for the metric in normal coordinates, ${}^*g_{\alpha\beta}$ (eg Veblen 1927):

$${}^*g_{\alpha\beta} = ({}^*g_{\alpha\beta})_0 + \tfrac{1}{3}({}^*R_{\alpha\beta\gamma\delta})_0 y^\beta y^\delta + O(y^3)$$

from which everything else follows easily. Thus, referring to equation (17), dq' is replaced by

$${}^*g^{1/2}\Pi\,dy^\alpha = ({}^*g^{1/2})_0(1-\tfrac{1}{6}({}^*R_{\alpha\beta})_0 y^\alpha y^\beta + \ldots)\Pi\,dy^\gamma \tag{24}$$

and Ω equals $\frac{1}{2}({}^*g_{\alpha\beta})_0 y^\alpha y^\beta$. In this case $\psi(y)$ is expanded in an ordinary Taylor series in y and Schrödinger's equation appears in the invariant form

$$-i\frac{\partial\psi}{\partial t} = \frac{1}{2}({}^*g^{\alpha\beta})_0\left(\frac{\partial^2\psi}{\partial y^\alpha\,\partial y^\beta}\right)_0 \tag{25}$$

which, in general coordinates, is just (1). Expansion (24) for ${}^*g^{1/2}$ is just that for Δ^{-1} (equation (21)). (If higher order terms had been needed then probably Walker's (1942) method would have been superior.)

Equation (25) introduces us to derivatives of ψ with respect to y evaluated at $y = 0$. These are, by definition, the 'extensions' of ψ referred to in § 3 and now is the appropriate time to say that the extensions of a scalar are just the symmetrized covariant derivatives. This allows us to rewrite the expansion (13) in a form more in keeping with modern differential calculus. We introduce the r-legs λ_a^α, which are such that

$$g^{\alpha\beta} = \lambda_a^\alpha \lambda_b^\beta g^{ab}$$

where the g^{ab} are constants, say $g^{ab} = \eta^{ab}$ with η diagonal elements ± 1 only, and then project everything onto these legs. In other words we go over to an anholonomic coordinate system (see Schouten 1954). Thus we define

$$\Omega^{a''} = \lambda_\alpha^{''a} \Omega^{\alpha''}$$

and the anholonomic (or directional) covariant derivative

$$\phi_{:ab\ldots c} = \lambda_a^\alpha \lambda_b^\beta \ldots \lambda_c^\gamma \phi_{''\alpha\beta\ldots\gamma}.$$

Then the expansion reads

$$\phi' = \phi'' - \Omega^{a''}\phi''_{:a} + \tfrac{1}{2}\Omega^{a''}\Omega^{b''}\phi''_{:ab} - \ldots$$

which is genuinely invariant under general coordinate transformations.

A particularly interesting case is when the Riemann space is a group manifold. The expansion is then well known and is

$$\phi(q') = \exp(\xi^a X_a'')\phi(q'') = \phi'' + \xi^a X_a'' \phi'' + \tfrac{1}{2}\xi^a \xi^b X_a'' X_b'' \phi'' + \ldots$$

where the ξ^a are the canonical coordinates of the group element $q'q''^{-1}$. In fact, $\xi^a = -\Omega^{a''}$ and we have the equality

$$\phi_{:(ab\ldots c)} = X_{(a} X_b \ldots X_{c)} \phi$$

where the X_a are the generators of the group.

These considerations may seem superfluous but if it is desired to discuss fields other than scalar ones (cf Ito 1963) it is virtually imperative to use an appropriate and covariant formalism and the more aspects of this that can be displayed the better.

We now wish to turn to another aspect of our calculation. Let us consider the general situation in which we do not specify the actual form of $\lim_{\epsilon\to 0}\langle q'', t' + \epsilon | q', t'\rangle$. We only say that $\langle q'', t''|q', t'\rangle$ is a biscalar. Then we can define a tensor drift μ^α and a tensor variance $\beta^{\alpha\beta}$ by

$$\mu_1^\alpha(q'', t'') = \lim_{\epsilon\to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q' \Omega^{\alpha''}(q'', q') \langle q'', t''|q', t'' - \epsilon\rangle$$

$$\beta_1^{\alpha\beta}(q'', t'') = \lim_{\epsilon\to 0} (\mathrm{i}\epsilon)^{-1} \int \mathrm{d}q' \Omega^{\alpha''}(q'', q') \Omega^{\beta''}(q'', q') \langle q'', t''|q', t'' - \epsilon\rangle,$$

and similar expressions for final averages over q''. In contrast to the a^α of equation (8) (ie the usual Kolmogorov mean) the μ^α is a vector simply because $\Omega^{\alpha''}(q'', q')$ is a vector at q'' and a scalar at q'. The corresponding Schrödinger equation is now found precisely as in the preceding section. It occurs immediately in the manifestly covariant form

$$\mathrm{i}\frac{\partial \psi''}{\partial t''} = \mu_1^{''\alpha}\psi''_{''\alpha} - \tfrac{1}{2}\beta_1^{''\alpha\beta}\psi''_{''\alpha\beta}.$$

Of course the equation obtained by the standard (non-covariant) method, (9), is equivalent to this one, the relation between the parameters being

$$a_{\mathrm{I}}^{\alpha} = \mu_{\mathrm{I}}^{\alpha} + \tfrac{1}{2}\beta_{\mathrm{I}}^{\gamma\beta}\Gamma_{\gamma\beta}^{\alpha}$$
$$b_{\mathrm{I}}^{\alpha\beta} = \beta_{\mathrm{I}}^{\alpha\beta}.$$

An alternative presentation of the standard method is that using Ito's theorems on stochastic differentials (Ito 1950, theorem 3.2, 1951, McKean 1969). Thus the stochastic differential equation,

$$\mathrm{d}q^{\alpha} = \mathrm{i}a_{\mathrm{I}}^{\alpha}(q, t)\,\mathrm{d}t + \sigma_{\mathrm{I}a}^{\alpha}(q, t)\,\mathrm{d}x^{a}, \tag{26}$$

where $x^{a}(t)$ is a standard 'Feynman–Wiener process', defines a quantum-mechanical stochastic process $q^{\alpha}(t)$ of mean a_{I}^{α} and variance $b_{\mathrm{I}}^{\alpha\beta} = \beta_{\mathrm{I}}^{\alpha\beta} = \sigma_{\mathrm{I}a}^{\alpha}\sigma_{\mathrm{I}b}^{\beta}g^{ab}$. Roughly speaking $\mathrm{d}x^{a}\,\mathrm{d}x^{b} \simeq \mathrm{i}g^{ab}\,\mathrm{d}t$ and $\mathrm{d}q^{\alpha}\,\mathrm{d}q^{\beta} \simeq \mathrm{i}\beta_{\mathrm{I}}^{\alpha\beta}\,\mathrm{d}t$.

Equation (26) is really a shorthand way of writing the stochastic integral equation

$$q''^{\alpha} - q'^{\alpha} = q^{\alpha}(t'') - q^{\alpha}(t') = \int_{t'}^{t''} \mathrm{i}\,\mathrm{d}t a_{\mathrm{I}}^{\alpha}(q(t), t) + \int_{t'}^{t''} \mathrm{d}x^{a}(t)\sigma_{\mathrm{I}a}^{\alpha}(q(t), t)$$

and this shows immediately why the calculation is non-covariant. The left-hand side is not a vector. Clearly, in order to make the Ito formalism covariant, one should use the world function $\Omega(q'', q')$ for the stochastic process $q(t)$ in a systematic way. In other words q'' and q' are two points on a stochastic process $q(t)$ and instead of the integral equation giving $(q''^{\alpha} - q'^{\alpha})$ we ought to have $\Omega^{\alpha''}(q'', q')$. More precisely, consider

$$g^{\alpha\beta}(q)\frac{\partial\Omega(q, q')}{\partial q^{\beta}} = \Omega^{\alpha}(q, q')$$

as a function of $q(t)$ and apply Ito's theorem to it. We find

$$\Omega^{\alpha''}(q'', q') = \int_{t'}^{t''} \mathrm{i}\,\mathrm{d}t[a_{\mathrm{I}}^{\beta}(q)\,\partial_{\beta}\Omega^{\alpha}(q, q') + \tfrac{1}{2}\sigma_{\mathrm{I}b}^{\beta}(q)\sigma_{\mathrm{I}c}^{\gamma}(q)g^{bc}\,\partial_{\beta}\,\partial_{\gamma}\Omega^{\alpha}(q, q')]$$
$$+ \int_{t'}^{t''} \mathrm{d}x^{a}\sigma_{\mathrm{I}a}^{\beta}(q)\,\partial_{\beta}\Omega^{\alpha}(q, q').$$

The equivalent differential form is found by letting t' tend to t'' and we find, on using the coincidence limit

$$\lim_{q' \to q''} \partial_{\beta}''\,\partial_{\gamma}''\Omega^{\alpha''}(q'', q') = -\Gamma_{\beta\gamma}^{\alpha}(q''),$$

that

$$\begin{aligned}\mathrm{d}\Omega^{\alpha''} &\equiv \lim_{t' \to t''} \Omega^{\alpha''}(q'', q') \\ &= \mathrm{i}[a_{\mathrm{I}}^{\alpha}(q'') - \tfrac{1}{2}\beta_{\mathrm{I}}^{\beta\gamma}(q'')\Gamma_{\beta\gamma}^{\alpha}(q'')]\,\mathrm{d}t + \sigma_{\mathrm{I}a}^{\alpha}(q'')\,\mathrm{d}x^{a} \\ &= \mathrm{i}\mu_{\mathrm{I}}''^{\alpha}\,\mathrm{d}t + \sigma_{\mathrm{I}a}''^{\alpha}\,\mathrm{d}x^{a}\end{aligned} \tag{27}$$

which is a covariant differential equation, in contrast to (26).

The same result can be obtained in reverse, so to speak, by writing down the definition of normal coordinates, y^{α}, ie

$$q'^{\alpha} = q''^{\alpha} + y^{\alpha} - \tfrac{1}{2}\Gamma_{\beta\gamma}^{\alpha}(q'')y^{\beta}y^{\gamma} + \mathrm{O}(y^{3})$$

(Schouten 1954) and then letting q' tend to q'' whilst keeping the term of second order in y. If we assume a general form for dy^α, like that of (26), we easily find for the parameters those values given in (27), allowing for a minus sign, $y^\alpha = -\Omega^{\alpha''}$ (Ruse 1931).

For the special values of the parameters, (11), the vector μ^α is zero and the set of vectors σ_a^α can be taken as the r-legs:

$$\sigma_a^\alpha = \lambda_a^\alpha,$$

and equation (27) takes the particularly simple form

$$d\Omega^{\alpha''} = \lambda_a^\alpha(q'')\, dx^a$$

which shows that the most natural 'diffusion' on the manifold is given by the most natural injection of a standard 'diffusion' on the tangent spaces (cf Gangolli 1964).

This is all we wish to say about the Ito formalism except to note that it is not strictly necessary to use a mid-point expansion of a stochastic integral in order to achieve a covariant diffusion equation. This modification of the Ito differential is due to Stratonovich (1968) (see also the useful remarks by McLaughlin and Schulman 1971). Further comments on this and other aspects of the stochastic differential must be reserved for another time.

6. Conclusion

We do not have a great deal to say in conclusion, having exhausted ourselves in the previous section. Perhaps we can content ourselves with voicing the opinion that re-casting the theory in the language of modern, coordinate-free differential geometry should prove rewarding, particularly the formalism of stochastic differentials.

Extension to the field theory domain presents little difficulty.

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